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A BRIEF ACCOUNT OF H. GRASSMANN'S GEOMETRICAL THEORIES.

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[CONTINUED FROM VOL. II, PAGE 11.]

THE SYSTEM OF THE PLANE.

XVIII. A plane is generated by the *simple* motion of a straight line. Now, in general, a straight line in moving may change either its position (translation), or its direction (rotation), or both position and direction (twist), thus generating a cylinder, a cone, or a scroll, respectively. The scroll degenerates into a plane only when the twist reduces to mere translation or rotation. Cylindrical and conical surfaces reduce to planes if the direction of the motion is guided by a straight line. The plane then is generated either by the translation of a line one of whose points moves along another straight line, or by a rotating line which in all its positions meets another straight line.

Just as a line has two opposite directions corresponding to the two possible directions of a point in describing it, so a plane presents two opposite *sides* corresponding to the two possible directions of motion of the generating line: a rotation, for instance, which from one side of the plane appears as right-handed or clockwise, appears as left-handed or counter-clockwise if looked at from the other side of the plane.

In the case of the plane, as in that of the straight line, the chief characteristic of the generating motion is its *simplicity*, that is, the property of being completely determined by two positions of the generating element.

XIX. If A, B, C are three points, any other point X in their plane may be expressed in the form

$$X = \alpha A + \beta B + \gamma C, \text{ where } \alpha + \beta + \gamma = 1.$$

Indeed, if the line CX meet AB in C', this point C' may be expressed in he

form $C' = \alpha' A + \beta' B$, where $\alpha' + \beta' = 1$; and then X may be determined through C and C', thus :—

$$X = \gamma' C' + \gamma C, \text{ where } \gamma + \gamma' = 1.$$

Hence, substituting, $$X = \alpha'\gamma' A + \beta'\gamma' B + \gamma C,$$

where $\alpha'\gamma' + \beta'\gamma' + \gamma = \alpha' - \alpha'\gamma + \beta' - \beta'\gamma + \gamma = 1 - \gamma + \gamma = 1.$

To find, by construction, the point $X = \alpha A + \beta B + \gamma C$, where $\alpha + \beta + \gamma = 1$, we may externally multiply the expression by any one of the given points, say A:

$$AX = \beta AB + \gamma AC,$$

which may be written (see Art. 16)

$$A(X - A) = \beta A(B - A) + \gamma A(C - A),$$

or $$X - A = \beta(B - A) + \gamma(C - A),$$

an expression for the vector to the point X.

XX. Let now e_1, e_2, e_3 be three unit-points; then, as appears from the preceding article, any point αe in the same plane can be expressed in the form

$$\alpha e = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3, \text{ where } \alpha_1 + \alpha_2 + \alpha_3 = \alpha;$$

and, *vice versa*, this expression always represents a point in the plane $e_1 e_2 e_3$ unless the co-efficient α of e vanishes. In this latter case we have $\alpha = \alpha_1 + \alpha_2 + \alpha_3 = 0$, hence

$$\alpha e = \alpha_1(e_1 - e_3) + \alpha_2(e_2 - e_3),$$

i. e. a *vector*, or, as we may also say, a point at infinity.* It is evident that any vector in the plane $e_1 e_2 e_3$ and hence any of its points at infinity, may be expressed in the form $\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$, where $\alpha_1 + \alpha_2 + \alpha_3 = 0$.

Hence, in general, *a homogeneous linear function of three points represents a point or a vector in the plane of the three points according as the sum of their co-efficients is different from or equal to zero.* On the other hand, any magnitude of the first degree (multiple point or vector) contained in the plane may be expressed as a linear homogeneous function of any three points of the plane. The plane is, therefore, *a system of the third degree.* Any four magnitudes of the first degree in such a system are connected by a homogeneous linear equation. Indeed, let

*The addition and subtraction of vectors is here supposed as known, as it is practically not different from what is to be found in all text-books on quaternions. Indeed, neither Hamilton nor Grassmann professes to have originated the so-called *geometrical* addition of lines, which before their time had been suggested first by the theory of imaginaries and later by mechanical considerations.

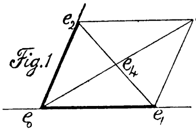

Fig.1

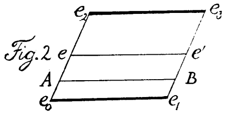

Fig.2

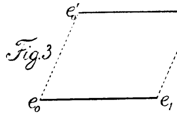

Fig.3

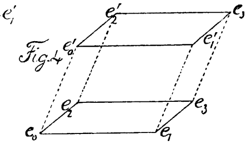

Fig.4

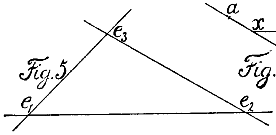

Fig.5

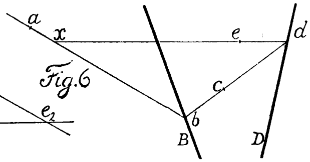

Fig.6

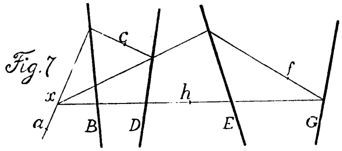

Fig.7

$$a = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3, \quad c = \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3,$$
$$b = \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3, \quad d = \delta_1 e_1 + \delta_2 e_2 + \delta_3 e_3$$

be the four magnitudes in the plane $e_1 e_2 e_3$. We can eliminate e_1, e_2, e_3 from the four equations and obtain the relation

$$\begin{vmatrix} a & \alpha_1 & \alpha_2 & \alpha_3 \\ b & \beta_1 & \beta_2 & \beta_3 \\ c & \gamma_1 & \gamma_2 & \gamma_3 \\ d & \delta_1 & \delta_2 & \delta_3 \end{vmatrix} = 0,$$

which is homogeneous and linear in a, b, c, d, any one of which may be expressed linearly by the three others.

XXI. The addition and substraction of *segments* in a plane is effected precisely as is the composition of forces in mechanics. We have (Fig. 1)

$$e_0 e_1 + e_0 e_2 = e_0 (e_1 + e_2),$$

or, since $e_1 + e_2 = 2e_4$, and $2e_0 e_4 = e_0 e_3$,

$$e_0 e_1 + e_0 e_2 = e_0 e_3.$$

This construction fails in the case of two parallel segments. We then have

(Fig. 2) $$e_0 e_1 + e_2 e_3 = e_0 (e_1 - e_0) + e_2 (e_3 - e_2) = (e_0 + e_2)(e_1 - e_0)$$
$$= 2e (e_1 - e_0) = 2e (e' - e) = 2ee',$$

where e, e' are the middle points respectively of $e_0 e_2$ and $e_1 e_3$.

Similarly, if

$$A = \alpha_0 e_0 + \alpha_2 e_2, \quad B = \alpha_0 e_1 + \alpha_2 e_3, \quad \therefore B - A = e_1 - e_0 = e_3 - e_2,$$

we have $$A(B - A) = \alpha_0 e_0 (e_1 - e_0) + \alpha_2 e_2 (e_3 - e_2) = \alpha_0 e_0 e_1 + \alpha_2 e_2 e_3.$$

Hence *any segment may be expressed as a sum of multiples of any two segments equal and parallel to it.*

This derivation of a segment from two parallel segments is strictly analogous to the derivation of a point from two given points, in the system of the line, by means of the formula $A = \alpha_0 e_0 + \alpha_2 e_2$. In both cases, the quotient $\alpha_2 : \alpha_0$ indicates the ratio in which the distance between the fixed elements is divided. The particular value -1 of this ratio requires special attention. Just as $e_2 - e_0$ was interpreted as representing the "step" from e_0 to e_2, or the amount of motion necessary to move the point e_0 in the simplest way into the position e_2, so here the difference $e_2 e_3 - e_0 e_1$ of two equal and parallel segments may be regarded as the space passed through by the segment $e_0 e_1$ if moved in the simplest way into the position $e_2 e_3$; that is, this difference may be said to represent the *parallelogram* $e_0 e_1 e_3 e_2$.

This parallelogram is affected with sign, since the generating motion can be performed in two opposite directions. It changes its sign whenever the two segments are interchanged, and also if both segments are reversed, conditions which are in accordance with our notation as a difference:

$$e_2e_3 - e_0e_1 = -(e_0e_1 - e_2e_3) = -(e_3e_2 - e_1e_0) = e_1e_0 - e_3e_2.$$

XXII. *The external product of two magnitudes of the first degree* a and b, viz:—

$$ab = (\alpha_1e_1 + \alpha_2e_2 + \alpha_3e_3)(\beta_1e_1 + \beta_2e_2 + \beta_3e_3)$$

$$= \begin{vmatrix} \alpha_2 & \alpha_3 \\ \beta_2 & \beta_3 \end{vmatrix} e_2e_3 + \begin{vmatrix} \alpha_3 & \alpha_1 \\ \beta_3 & \beta_1 \end{vmatrix} e_3e_1 + \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} e_1e_2,$$

is a sum of multiple segments, and hence either a segment or a parallelogram.

Indeed if a and b are both points, we may take $a = \alpha_1e_1$, $b = \alpha_2e_2$; hence $ab = \alpha_1\alpha_2 \,.\, e_1e_2$, a segment.

If one of the factors is a point while the other is a vector, the latter may always be assumed to pass through the point, and we may take $a = \alpha_1e_1$, $b = \alpha_2(e_2 - e_1)$; hence $ab = \alpha_1\alpha_2e_1(e_2 - e_1) = \alpha_1\alpha_2 \,.\, e_1e_2$, a segment.

If, however, both a and b be vectors so that we have

$$a = \alpha_1(e_1 - e_0), \quad b = \alpha_2(e_2 - e_0),$$

we obtain

$$ab = \alpha_1\alpha_2\left[e_1(e_2 - e_0) - e_0e_2\right],$$

or, since $e_1(e_2 - e_0) = e_1e_3$ represents a segment passing through e_2 and equal in length and direction to the vector $e_2 - e_0$ (see Art. 16),

$$ab = \alpha_1\alpha_2(e_1e_3 - e_0e_2),$$

a parallelogram.

Segments and parallelograms may be called the *magnitudes of the second degree* in the system of the plane. The general expression of such a magnitude is

$$a = \alpha_1e_0e_1 + \alpha_2e_2e_3, \quad \text{or } a = \alpha_1\varepsilon_1 + \alpha_2\varepsilon_2,$$

where $\varepsilon_1 = e_0e_1$ and $\varepsilon_2 = e_2e_3$, which are any two segments in the plane, may be regarded as *units of the second degree*.

The relations between magnitudes of the second degree in the plane are analogous to those between magnitudes of the first degree in the system of the line (compare Arts. 12 and 13).

XXIII. The *external product* $e_1e_2e_3$ *of three unit-points* is called a *plane segment* (*Flächentheil*). It is a magnitude of the third degree, and we interpret it as representing a portion of the plane $e_1e_2e_3$ equal in area to twice the triangle contained between the three points. This plane segment $e_1e_2e_3$ is distinguished from the parallelogram $e_2e_3 - e_0e_1$ (where $e_1 - e_0 = e_3 - e_2$) in a way similar to that in which the linear segment e_0e_1 is distinguished from the vector $e_1 - e_0$. The seg-

ment e_0e_1, being confined to the line determined by the points e_0 and e_1, is a localized vector; the plane segment $e_1e_2e_3$ is a localized parallelogram, confined to the plane determined by the points e_1, e_2, e_3. While two parallelograms of equal area and sign are equal whenever their planes are parallel, two plane segments of the same area and sign are considered equal only if they lie in the same plane.

The vector $e_1 - e_0$ may be regarded as the difference of position of two points, that is, as a scalar quantity, having direction, but without any definite position in space. This quantity $e_1 - e_0$ if applied as co-efficient to a point e_0' (Fig. 3) forms a segment $e_0'(e_1 - e_0) = e_0'(e_1' - e_0') = e_0'e_1'$, or an extensive magnitude of one dimension, the numerical value, direction, and sign of which are the same as those of the vector, while its position is fixed through the point e_0'.

Similarly the parallelogram $e_2e_3 - e_0e_1 = \varepsilon_2 - \varepsilon_1$, was above defined as the difference of position of two parallel unit-segments; it is a scalar quantity possessing the common "aspect" of a system of parallel planes. If it be applied as co-efficient to a point e_0', we find (Fig. 4)

$$e_0'(e_2e_3 - e_0e_1) = e_0'(e_2'e_3' - e_0'e_1') = e_0'e_2'e_3',$$

a plane segment, i. e. an extensive magnitude of two dimensions.

XXIV. *The external product of three magnitudes of the first degree*

$$\begin{aligned} a &= \alpha_1e_1 + \alpha_2e_2 + \alpha_3e_3, \\ b &= \beta_1e_1 + \beta_2e_2 + \beta_3e_3, \\ c &= \gamma_1e_1 + \gamma_2e_2 + \gamma_3e_3, \end{aligned}$$

if formed according to the ordinary rules of algebra, is a sum of twenty-seven terms of the form $\alpha_i\beta_k\gamma_le_ie_ke_l$. To reduce it to a simpler expression it becomes necessary to define the operation $(e_ie_k)\,e_l$, i. e. to establish a rule for multiplying products. We shall assume the associative law of algebra to hold in this case, i. e. we assume

$$(e_ie_k)\,e_l = e_i\,(e_ke_l) = e_ie_ke_l.$$

This law in connection with the two fundamental laws of external multiplication,

$$e_ie_k = -e_ke_i, \quad e_ie_i = 0,$$

will enable us to reduce all those products $e_ie_ke_l$ in the above sum which do not vanish to either of the forms $\pm e_1e_2e_3$, and we readily find

$$abc = \Sigma \pm \alpha\beta\gamma\,.\,e_1e_2e_3 = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} e_1e_2e_3.$$

Hence *the external product of three magnitudes of the first degree is a multiple plane segment.*

It vanishes only when $\Sigma \pm \alpha\beta\gamma = 0$, i. e. when the three magnitudes a, b, c are not independent of one another.

If a, b, c are all three vectors, they are always connected by such a relation between their co-efficients. Indeed, we then have

$$\alpha_1 + \alpha_2 + \alpha_3 = 0, \quad \beta_1 + \beta_2 + \beta_3 = 0, \quad \gamma_1 + \gamma_2 + \gamma_3 = 0;$$

$$\therefore \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} = \begin{vmatrix} \alpha_1 + \alpha_2 + \alpha_3 & \alpha_2 & \alpha_3 \\ \beta_1 + \beta_2 + \beta_3 & \beta_2 & \beta_3 \\ \gamma_1 + \gamma_2 + \gamma_3 & \gamma_2 & \gamma_3 \end{vmatrix} = \begin{vmatrix} 0 & \alpha_2 & \alpha_3 \\ 0 & \beta_2 & \beta_3 \\ 0 & \gamma_2 & \gamma_3 \end{vmatrix} = 0.$$

The product of three complanar vectors always vanishes.

If a, b, c are not all three vectors, a relation between their co-efficients of the form $\Sigma \pm \alpha\beta\gamma = 0$ indicates that the three magnitudes lie in the same straight line.

XXV. Just as, in the system of the line, by assuming a linear segment $e_1e_2 = 1$, we were enabled to express any other segment of the same line as a scalar quantity, so in the plane we may put $e_1e_2e_3 = 1$ and may then express any magnitude of the third degree in the same plane as a scalar, the plane segment $e_1e_2e_3$ being the geometrical unit for these magnitudes.

Again, remembering the rule that in an external product the first factor is the multiplicand and the second the multiplier, we call e_2e_3 the *complement* of e_1, in symbols $e_2e_3 = | e_1$; and this is really nothing but another way of stating the relation $e_1e_2e_3 = 1$, just as in ordinary algebra the expression $a = \frac{1}{b}$ is equivalent to the relation $ab = 1$.

We have thus:—

$$\begin{aligned} e_2e_3 &= | e_1, \text{ since } e_1 . e_2e_3 = 1 \text{ (read: } e_1 \text{ multiplied by } e_2e_3 \text{ equals } 1), \\ e_3e_1 &= | e_2, \text{ since } e_2 . e_3e_1 = - e_2e_1e_3 = + e_1e_2e_3 = 1, \\ e_1e_2 &= | e_3, \text{ since } e_3 . e_1e_2 = - e_1e_3e_2 = + e_1e_2e_3 = 1. \end{aligned}$$

Also:

$$\begin{aligned} e_1 &= | e_2e_3, \text{ since } e_2e_3 . e_1 = - e_2e_1e_3 = + e_1e_2e_3 = 1, \\ e_2 &= | e_3e_1, \text{ since } e_3e_1 . e_2 = - e_1e_3e_2 = + e_1e_2e_3 = 1, \\ e_3 &= | e_1e_2, \text{ since } e_1e_2 . e_3 = 1. \end{aligned}$$

Finally we may write

$$\begin{aligned} 1 &= | e_1e_2e_3, \text{ since } e_1e_2e_3 . 1 = e_1e_2e_3 = 1, \\ e_1e_2e_3 &= | 1, \quad \text{ since } 1 . e_1e_2e_3 = 1. \end{aligned}$$

It follows at once from these formulæ that

$$| e_1 = || e_2e_3 = e_2e_3,$$

i.e. *the complement of the complement of a magnitude is the original magnitude.* It

is to be observed that this rule is different from what we found in the system of the line where from $e_1e_2 = 1$ we deduced $e_2 = \mid e_1$ and $\| e_1 = - e_1$ (see Art. 17).

XXVI. The introduction of this idea of the complement of a factor is of special value as furnishing an interpretation, within the system of the plane, of products involving more than three unit-factors. Such products could otherwise be interpreted only as magnitudes of a degree higher than the third and would therefore not find any place in the system of the plane.

Thus, for instance, the product of two linear segments in the same plane, $e_1e_2 \,.\, e_2e_3$, may be written $\mid e_3 \,.\, \mid e_1$; and if in analogy with the law of ordinary algebra, $a \,.\, b = \dfrac{1}{\frac{1}{a} \cdot \frac{1}{b}}$, we assume for our multiplication the analogous law, $a \,.\, b = \mid (\mid a \,.\, \mid b)$, we obtain

$$e_1e_2 \,.\, e_2e_3 = \mid (\mid e_1e_2 \,.\, \mid e_2e_3) = \mid (e_3 \,.\, e_1) = \mid e_3e_1 = e_2,$$

or, since $e_1e_2e_3 = 1$,

$$e_1e_2 \,.\, e_2e_3 = e_1e_2e_3 \,.\, e_2,$$

i. e. *the product of two linear segments in the same plane is their point of intersection.*

Similarly we find (Fig. 5)

$$\begin{aligned} e_3e_1 \,.\, e_1e_2 &= e_1e_2e_3 \,.\, e_1, \\ e_1e_2 \,.\, e_2e_3 &= e_1e_2e_3 \,.\, e_2, \\ e_2e_3 \,.\, e_3e_1 &= e_1e_2e_3 \,.\, e_3; \end{aligned}$$

multiplying these three equations we obtain

$$(e_2e_3 \,.\, e_3e_1 \,.\, e_1e_2)^2 = (e_1e_2e_3)^4,$$

or

$$e_2e_3 \,.\, e_3e_1 \,.\, e_1e_2 = (e_1e_2e_3)^2,$$

i. e. the product of three segments forming a triangle is equal to the square of the double area of this triangle. This product will therefore vanish only when the area of the triangle is equal to zero, i. e. when the three lines e_2e_3, e_3e_1, e_1e_2 pass through the same point. And, *vice versa*, whenever three such lines pass through the same point, we have the relation $e_2e_3 \,.\, e_3e_1 \,.\, e_1e_2 = 0$.

To extend these rules of multiplication to magnitudes of the first degree in general, we have only to define the complement of such a magnitude

$$a = a_1e_1 + a_2e_2 + a_3e_3;$$

and this is done by the formula

$$\mid a = a_1 \mid e_1 + a_2 \mid e_2 + a_3 \mid e_3.$$

It follows that if

$$a = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3, \quad b = \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3,$$

$$\therefore\ ab = \begin{vmatrix} \alpha_2 & \alpha_3 \\ \beta_2 & \beta_3 \end{vmatrix} e_2 e_3 + \begin{vmatrix} \alpha_3 & \alpha_1 \\ \beta_3 & \beta_1 \end{vmatrix} e_3 e_1 + \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} e_1 e_2,$$

then $\qquad | \, a = \alpha_1 e_2 e_3 + \alpha_2 e_3 e_1 + \alpha_3 e_1 e_2, \quad | \, b = \beta_1 e_2 e_3 + \beta_2 e_3 e_1 + \beta_3 e_1 e_2,$

and $\quad | \, a \, . \, | \, b = \begin{vmatrix} \alpha_2 & \alpha_3 \\ \beta_2 & \beta_3 \end{vmatrix} e_1 + \begin{vmatrix} \alpha_3 & \alpha_1 \\ \beta_3 & \beta_1 \end{vmatrix} e_2 + \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} e_3, \quad \therefore \, | \, (ab) = | \, a \, . \, | \, b.$

XXVII. Any magnitude in the system of the plane may be expressed by means of three fundamental unit-points e_1, e_2, e_3; the double area of the triangle contained between these points being assumed as the geometrical unit for extensive magnitudes of two dimensions. The product of any two magnitudes of the first degree in e_1, e_2, e_3 is a magnitude of the second degree; the product of three magnitudes of the first degree, or of one magnitude of the first and one of the second, is a magnitude of the third degree. The degree thus being increased by multiplication, the product is called by Grassmann a *progressive* product.

When, however, the sum of the degrees of the factors exceeds three (i. e. the degree of the system of the plane), the degree of the resulting magnitude is less than the degree of the original magnitudes. The product of two linear segments, for instance, is their point of intersection. In this case Grassmann calls the product *regressive*.

In many geometrical investigations the value of the scalar coefficients which make the point an extensive magnitude of the first degree, and the line a linear segment of definite magnitude, are immaterial and may be disregarded. In this case we may say simply that the product of two points is the line connecting them; the product of two lines is their point of intersection; the product of a line and a point is a scalar which becomes zero when the point lies on the line. Any construction by means of the ruler alone, that is, requiring nothing but intersecting of lines and joining of points, may be expressed in the form of a continued product. Thus, if small letters be used to designate points and capitals be used to designate lines, the product $abCdEf$ expresses the following construction: Join the points a and b by a line ab; the intersection of this line with the line C gives a point abC which joined to the point d determines a line $abCd$ cutting the line E in a point $abCdE$; the line which joins this point to the point f is represented by $abCdEf$ Grassmann calls these expressions *planimetric* products

XXVIII. If the resulting line were subject to the condition of passing through a given point g, we might express this simply by writing $abCdEfg = 0$; and the condition that a resulting point should lie in a given line would be

expressed in a similar way by equating a planimetric product to zero. If one of the points or lines used in the construction were not given, such an equation would evidently represent the geometrical locus of the point, or the envelope of the line, represented by the product.

Thus, for instance, we obtain the equation of a conic section by expressing the condition that the opposite sides of a hexagon one of whose vertices is variable intersect in three points situated in a straight line. Let a, b, c, d, e, x be any six points, of which the first five are given; the condition that x is a point of the conic determined by a, b, c, d, e, may be expressed by the equation

$$(ab \,.\, de)\,(bc \,.\, ex)\,(cd \,.\, xa) = 0.$$

Newton's "organic" description of conics leads to another form of this equation. Let the three sides of a triangle xbd turn about three fixed points a, c, e, while two of its vertices, b and d, move along fixed lines B, D; then the third vertex x will describe the conic whose equation is

$$axBcDxe = 0.$$

The equation shows that a, e, BD, acD, ecB are points of the curve; for the substitution of these symbols for x makes the equation vanish identically.

A valuable investigation of certain properties of the "mystic hexagram" by means of Grassmann's methods was given by F. Caspary, in Crelle's *Journal*, Vol. XCII (1882), pp. 123–144.

XXIX. The most important property of such an equation is that it indicates, by the number of times the variable appears in it, the order or class of the locus or envelope it represents. For a rigorous proof of this proposition I must refer to the works of Grassmann and Schlegel. I can here only briefly indicate the application of the method to higher curves.

The equation $axBcDxEfGxh = 0$, which contains the variable point three times, represents a curve of the third order. The following linear construction of this curve may be directly read off from the equation: Let x be the common vertex of two contiguous triangles whose bases and not-common sides turn about the fixed points c, f, a, h, respectively, while the end-points of their bases move in the fixed lines B, D, E, G; then x describes a curve of the third order.

It is to be noticed that this is the first linear geometrical construction of the general cubic ever given. It was proposed by Grassmann in a paper published in Crelle's *Journal*, Vol. XXXI (1846), pp. 111–132.

The generalization of this method presents no difficulty. Indeed, let n triangles have a common vertex, the angles at this vertex being contiguous; then, if all the other vertices move in fixed straight lines, while the bases and the two

extreme sides turn about fixed points, the common vertex describes a curve of the n^{th} order, whose equation is

$$axB_1c_1D_1xB_2c_2D_2xB_3 \ . \ \ . \ \ . \ ex = 0.$$

Grassmann has also successfully applied his methods to the theory of projective ranges and pencils, and to the theory of harmonic poles and polars. In both cases was he led to noteworthy generalizations.

It is, however, in the science of rational mechanics that the application of Grassmann's theories and methods, in combination with those of Sir Wm. R. Hamilton, will probably prove of the greatest importance. As a valuable step in this direction, the reader might well be referred to the *Vector Analysis* of Professor J. Willard Gibbs (New Haven, 1881–1884), were it not for the fact that the words "not published" which appear on the title-page would seem to exclude that work from general circulation.

THE ATTRACTION OF A RIGHT CIRCULAR CYLINDER ON A PARTICLE.

By CHAS. H. KUMMELL, Washington, D. C.

I propose to give here a more elaborate treatment of this problem, which is one of the illustrations in my paper, read before the Mathematical Section of the Philosophical Society of Washington entitled: *Can the Attraction of a Finite Mass be Infinite?*

Let m = the mass of attracted particle,

ϑ = density of attracting cylinder,

r = radius of attracting cylinder,

h = height of attracting cylinder,

d = distance of attracted particle from axis of cylinder,

e = elevation of attracted particle above base of cylinder.

Assume the vertical line through the attracted particle for the axis of cylindrical co-ordinates in which ρ = horizontal radius vector, v = horizontal angle with central radius vector, z = elevation above (or depression below) attracted particle; then, since

$$\sqrt{(\rho^2 + z^2)} = \text{distance of any mass element from particle}$$